A Hierarchy of Immunity and Density for Sets of Reals

Takayuki Kihara*

Mathematical Institute, Tohoku University, Sendai 980-8578, Japan
kihara.takayuki.logic@gmail.com

Abstract. The notion of immunity is useful to classify degrees of non-computability. Meanwhile, the notion of immunity for topological spaces can be thought of as an opposite notion of density. Based on this viewpoint, we introduce a new degree-theoretic invariant called *layer density* which assigns a value n to each subset of Cantor space. Armed with this invariant, we shed light on an interaction between a hierarchy of density/immunity and a mechanism of type-two computability.

Keywords: computability theory, Π^0_1 class, Medvedev degree

1 Introduction

1.1 Summary

The study of *immunity* was initiated essentially by Post in 1944. Demuth-Kučera [5] studied the notion of immunity for closed sets in Baire space. Immunity for a closed set indicates that it is "*far from dense*". They showed that any 1-generic real computes no element of any immune co-c.e. closed set, and hence no 1-generic real computes a Martin-Löf random real. Binns [1] introduced many notions of *hyperimmunity* for closed sets to classify degrees of difficulty of co-c.e. closed sets. Cenzer-Kihara-Weber-Wu [4] started the systematic study on immunity for closed sets. Higuchi-Kihara [6] clarified that such notions indicating being "*nearly/far from dense*" are extremely useful to study a hierarchy of *nonuniform computability on sets of reals*. We investigate a hierarchy of properties that are "*nearly dense*", by introducing a new degree-theoretic invariant called *layer density* which assigns a value n to each subset of any computable metric space. In this way, we shed light on an interaction between a hierarchy of density and a mechanism of type-two computability. We also continue the work [6] on the structure *inside the Turing upward closure* of any co-c.e. closed set.

1.2 Notation and Convention

Much of our notation in this paper follows that in [6]. For basic terminology on Computability Theory and Computable Analysis, see [3, 8, 9]. For any sets X

* This work was supported by Grant-in-Aid for JSPS fellows.

and Y, f is said to be a *function from X to Y* if $\mathrm{dom}(f) \supseteq X$ and $\mathrm{range}(f) \subseteq Y$ hold. We use the symbol $^\frown$ for the concatenation. For $\sigma \in \omega^{<\omega}$, we let $|\sigma|$ denote the length of σ. Moreover, $f \upharpoonright n$ denotes the unique initial segment of f of length n. We also define $[\sigma] = \{f \in \omega^\omega : f \supset \sigma\}$. For a tree $T \subseteq \omega^{<\omega}$, let $[T]$ denote the set of all infinite paths through T. For a subset A of a space X, $\mathrm{cl}(A)$, and $\mathrm{ext}(A)$ denote the closure, and the exterior of A, respectively. A *representation* ρ of a space X is a surjection $\rho :\subseteq \omega^\omega \to X$. Let $\mathcal{A}_-(X)$ denote the hyperspace consisting of closed subsets of X represented by $\psi_- : \alpha \mapsto X \setminus \bigcup_n \beta_{\alpha(n)}$. Here, $\{\beta_n\}_{n\in\omega}$ is a fixed countable base of X. A computable element of $\mathcal{A}_-(X)$ (i.e., $\psi_-(\alpha)$ for some computable $\alpha \in \omega^\omega$) is called a *co-c.e. closed set* or a *Π^0_1 class.*

2 Computability with Layers

2.1 Density and Immunity

Let X be a topological space, and $\mathcal{B}$ be a collection of open sets in X. A subset $S \subseteq X$ is said to be *$\mathcal{B}$-dense* if it intersects with all nonempty open sets contained in $\mathcal{B}$. By restricting $\mathcal{B}$, one may introduce various "*pre-dense*" properties. For instance, *immunity* [4] and *hyperimmunity* [1] can be introduced in this way. A variety of interactions are known between density/immunity and degrees of difficulty [4, 5, 7]. To introduce nice $\mathcal{B}$-density notion, we consider the following effective notion for open sets: An open set $S \subseteq X$ is *bi-c.e. open* if both S and $\mathrm{ext}(S)$ are c.e. open. We fix $X = 2^\omega$. A sequence $\{B_n\}$ of open rational balls is *nontrivial* if it contains no empty set, and $\liminf_n \mathrm{diam}(B_n) = 0$; *computable* if it is uniformly computable (hence, $\bigcup_n B_n$ is c.e. open); and *decidable* if it is computable, and $\bigcup_n B_n$ is bi-c.e. open. Let $P \subseteq 2^\omega$ be a closed set, and let T_P^{ext} denote the tree $\{\sigma \in 2^{<\omega} : P \cap [\sigma] \neq \emptyset\}$. Cenzer et al. [4] introduced the following notion: P is *immune* if T_P^{ext} contains no infinite computable subset. P is *tree-immune* if T_P^{ext} contains no infinite computable subtree.

Proposition 1. *Let $P \subseteq 2^\omega$ be a closed set with no computable element. Then, P is not immune if and only if it is $\mathcal{B}$-dense for some nontrivial computable sequence $\mathcal{B}$ of open balls; P is not tree-immune if and only if it is $\mathcal{B}$-dense for some nontrivial decidable sequence $\mathcal{B}$ of pairwise disjoint open balls.*

Proof. Assume that P is $\mathcal{B}$-dense via an infinite computable sequence $\mathcal{B}$ of open balls. For each $B \in \mathcal{B}$, we choose the smallest clopen set $[\sigma]$ including B, and enumerate $[\sigma]$ into another sequence $\mathcal{B}^*$. As $\liminf_{B\in\mathcal{B}} \mathrm{diam}(B) = 0$, the sequence $\mathcal{B}^*$ is infinite. It is easy to see that P is also $\mathcal{B}^*$-dense. Therefore, P is not immune. Another direction is obvious.

Assume that P is not tree-immune via an infinite computable tree $V \subseteq T_P^{ext}$. As P has no computable element, V has infinitely many leaves, i.e., $L = \{\sigma \in V : (\forall i < 2)\ \sigma^\frown i \notin V\}$ is infinite. Then, we define $\mathcal{B} = \{[\sigma] : \sigma \in L\}$. To enumerate the exterior of $\bigcup \mathcal{B}$, for each $\sigma \in 2^{<\omega}$, we define $(\sigma^\frown i)^* = \sigma^\frown(1 - i)$ for each $i < 2$. Then, the exterior of $\bigcup \mathcal{B}$ is generated by the computable set $\{\sigma \in 2^{<\omega} \setminus V : \sigma^* \in V\}$, since $[V]$ has no interior. Hence, $\bigcup \mathcal{B}$ is bi-c.e. open.

Conversely, assume that P is $\mathcal{B}$-dense for a decidable sequence $\mathcal{B} = \{[\sigma_n]\}_{n\in\omega}$ of open balls. Then, there is a computable enumeration of all strings σ that are comparable with σ_n for some $n \in \omega$, since $\mathcal{B} = \{\sigma_n\}_{n\in\omega}$ is computable. Moreover, $[\sigma] \subseteq \mathrm{ext}(\bigcup \mathcal{B})$ if and only if there is no $n \in \omega$ such that σ is comparable with σ_n. Hence, the set U consisting of all strings $\sigma \in 2^{<\omega}$ which are comparable with some σ_n is computable, since $\mathrm{ext}(\bigcup \mathcal{B})$ is c.e. open. Then, we can compute the tree $V = \{\sigma \in 2^{<\omega} : (\exists n \in \omega)\ \sigma \subseteq \sigma_n\}$ as follows: If $\sigma \notin U$, then declare $\sigma \notin V$. If $\sigma \in U$, then σ must be comparable with some σ_n. Wait for the least such $n \in \omega$, and if $\sigma \subseteq \sigma_n$, then declare $\sigma \in V$. Otherwise, declare $\sigma \notin V$. This algorithm correctly computes V, since the sequence $\{\sigma_n\}_{n\in\omega}$ is pairwise incomparable. Then, for each $\sigma \subseteq \sigma_n$, the open ball $[\sigma] \supseteq [\sigma_n]$ intersects with P, by $\mathcal{B}$-density of P. □

By considering layers $\{B_j\}_{j\in\omega}, \{B_{j,k}\}_{j,k\in\omega}, \{B_{j,k,l}\}_{j,k,l\in\omega}, \dots$ of open balls hitting a set $P \subseteq 2^\omega$, we may strengthen the notion of $\mathcal{B}$-density. Here, it is required that P is $\{B_j\}_{j\in\omega}$-dense; $P \cap B_j$ is $\{B_{j,k}\}_{k\in\omega}$-dense for each $j \in \omega$; $P \cap B_j \cap B_{j,k}$ is $\{B_{j,k,l}\}_{l\in\omega}$-dense for each $j, k \in \omega, \dots$

Definition 1. *Let Y be a subset of $X = 2^\omega$.*

1. *A sequence $\{B_{n,m}\}_{(n,m)\in I\times J}$ of open balls is* an J-refinement of $\{A_n\}_{n\in I}$ in *Y if it is pairwise disjoint, and $B_{n,m} \subseteq A_n$ for any $(n, m) \in I \times J$.*
2. *A sequence $\{\mathcal{B}_k\}_{k<n}$ (resp. $\{\mathcal{B}_k\}_{k\in\omega}$) of decidable sequences of nonempty open rational balls is* an n-layer in Y *(resp.* an ∞-layer*) if $\mathcal{B}_{k+1} = \{B^{k+1}_{i,j}\}_{i,j}$ is an ω-refinement of $\mathcal{B}_k = \{B^k_i\}_i$ in Y, and $\{B^{k+1}_{i,j}\}_{j\in\omega}$ is decidable uniformly in i, for any $k < n - 1$ (resp. for any $k \in \omega$).*
3. *For $n \in \omega \cup \{\infty\}$, a set $P \subseteq X$ is* n-layered *if there is an n-layer $\mathfrak{B} = \{\mathcal{B}_k\}$ in P such that P is $\bigcup \mathfrak{B}$-dense, where $\mathcal{B}_0 = \{X\}$.*
4. *The* layer density *of a set $P \subseteq X$ is defined as follows:*

$$\mathrm{density}(P) = \sup\{n \in \omega \cup \{\infty\} : P \text{ is } n\text{-layered}\}.$$

Here, the ordering on $\omega \cup \{\omega, \infty\}$ is defined as $n < \omega < \infty$ for any $n \in \omega$.

Proposition 2. *Let P be a subset of 2^ω. Then, P is empty if and only if* $\mathrm{density}(P) = 0$*; If $Q \subseteq P$, then* $\mathrm{density}(Q) \leq \mathrm{density}(P)$*; If P is dense, then P is ∞-layered.* □

Proposition 3. *Let $P \subseteq 2^\omega$ be a closed set with no computable element. Then, $P \subseteq 2^\omega$ is n-layered if and only if there is a sequence $\{T_i\}_{i<n}$ of infinite computable trees such that $[T_n] \subseteq P$ for any $i < n$, and $T_i \subseteq T^{ext}_{i+1}$ for any $i < n - 1$.*

Proof. Assume that $P \subseteq 2^\omega$ has such a sequence $\{T_i\}_{i\leq n}$ of infinite computable trees. We effectively enumerate all leaves $\{\sigma^i_k\}_{k\in\omega}$ of the tree T_i, for each $i < n$. Then, as Proposition 1, $\{2^\omega, \{[\sigma^0_k]\}_{k\in\omega}, \dots, \{[\sigma^{n-1}_k]\}_{k\in\omega}\}$ forms an n-layer of P.

Conversely, assume that $P \subseteq 2^\omega$ is n-layered via $\{\mathcal{B}_i\}_{i\leq n}$. As in the proof of Proposition 1, without loss of generality, we may assume $\mathcal{B}_i$ is of the form $\{[\sigma^i_k]\}_{k\in\omega}$, for each $i \leq n$. Then, we define $T_i = \{\sigma \in 2^{<\omega} : (\exists k \in \omega)\ \sigma \subseteq \sigma^{i+1}_k\}$. We can see that T_i is computable for each $i < n$, as Proposition 1. Then, $\{T_0, T_1, \dots, T_{n-1}, T_P\}$ is the desired sequence. □

Example 1. Let P be a co-c.e. closed subset of 2^ω. Then, for a fixed computable tree T_P with $P = [T_P]$, we have the computable set $\{\rho_n\}_{n\in\omega}$ of all leaves of T_P. The *concatenation* $P^\frown P$ is defined by $\bigcup_n \rho_n{}^\frown P$. Consider $P^{(1)} = P$; $P^{(n+1)} = P^\frown P^{(n)}$; $P^{(\omega)} = \bigcup_n \rho_n{}^\frown P^{(n)}$; and $P^{(\infty)} = \bigcup_n P^{(n)}$. Then, density$(P^{(n)}) \geq n$; density$(P^{(\omega)}) \geq \omega$; and density$(P^{(\infty)}) = \infty$. See also Higuchi-Kihara [6].

2.2 Learnability on Topological Spaces

When we try to extract effective content in classical mathematics, we sometimes encounter the notion of nonuniform computability [2, 10]. The deep structures of subnotions of nonuniformly computability have been studied [6].

Definition 2 (Learnability). *Let X be a topological space with a representation $\theta :\subseteq \omega^\omega \to X$, and fix a new symbol $? \notin X$.*

1. *The* representation $\theta_?$ *of the space $X_? = X \cup \{?\}$ is defined as $\theta_?(\langle 0\rangle^\frown\alpha) = \theta(\alpha)$, and $\theta_?(\langle 1\rangle^\frown\alpha) = ?$, for any $\alpha \in \omega^\omega$.*
2. *A sequence $\{f_n\}_{n\in\omega}$ of partial functions $f_n :\subseteq Y \to X_?$ is* ?-good *if $? \in \{f_n(\alpha), f_{n+1}(\alpha)\}$ whenever $f_n(\alpha) \neq f_{n+1}(\alpha)$.*
3. *The* discrete limit *of a ?-good sequence $\{f_n\}_{n\in\omega}$ of partial functions $f_n :\subseteq Y \to X_?$ is a partial function $\lim_n f_n :\subseteq Y \to X$ defined as follows.*
$$\lim_n f_n(\alpha) = \begin{cases} f_t(\alpha), & \textit{if } (\forall s \geq t)\ f_s(\alpha) \neq ?, \\ \textit{undefined}, & \textit{if } (\exists^\infty s)\ f_s(\alpha) = ?. \end{cases}$$
4. *A function $f :\subseteq Y \to X$ is* learnable *if it is the discrete limit of a computable ?-good sequence $\{f_n\}_{n\in\omega}$ of partial functions $f_n :\subseteq Y \to X_?$.*
5. *An* anti-Popperian point *of a ?-good sequence $\{f_n\}_{n\in\omega}$ is a point $\alpha \in \omega^\omega$ such that $f_n(\alpha) = ?$ at most finitely many $n \in \omega$, but $\lim_n f_n(\alpha)$ is undefined.*
6. *A function $f : Y \to X$ is* eventually Popperian learnable *(abbreviated as* e.P. learnable*) if it is the discrete limit of a computable ?-good sequence $\{f_n\}_{n\in\omega}$ of partial functions $f_n :\subseteq Y \to X_?$ with no anti-Popperian points.*

Lemma 1 (Blum-Blum Locking). *Let (X, d) be a Polish space with a representation, and Q be a closed set in X. For every learnable function $\Gamma : Q \to P$, there is an open set $U \subseteq X$ such that $Q \cap U \neq \emptyset$, and the restriction $\Gamma|_U : Q \cap U \to P$ is computable.*

Proof. Suppose not. Fix a learnable function $\Gamma = \lim_s \Gamma_s : Q \to P$ witnessing the falsity of the assertion. Then, for any open set U_0^* and every $s_0 \in \omega$, there is $s_1 \geq s_0$ such that the open set $U_1 = \Gamma_{s_1}^{-1}\{?\}$ has a nonempty intersection with Q. Then U_1 contains an open ball $\{p \in X : d(p, q) < \varepsilon\}$ with $q \in Q$ and $\varepsilon > 0$. Pick $U_1^* = \{p \in X : d(p, q) < \min\{\varepsilon/2, 2^{-n}\}\} \subseteq U_1$. By iterating this procedure, we can get a decreasing sequence $\{U_n^*\}_{n\in\omega}$. Choose $x_n \in U_n^* \cap Q$. Then, $\{x_n\}_{n\in\omega}$ converges to an element $x \in Q \cap \bigcap_n \mathrm{cl}(U_n^*)$. By our choice of $\{U_n^*\}_{n\in\omega}$, we see that $\Gamma_s(x) = ?$ for infinitely many $s \in \omega$. Consequently, $\Gamma(x) = \lim_s \Gamma_s(x)$ is undefined, i.e., $\mathrm{dom}(\Gamma) \not\supseteq Q$. ☐

3 Degrees of Difficulty

3.1 Layer Density as a Degree-Theoretic Invariant

Theorem 1. *Let $P, Q \subseteq 2^\omega$ be co-c.e. closed sets with no computable element. If a computable function exists from P to Q, then* $\mathrm{density}(P) \leq \mathrm{density}(Q)$.

Proof. A sequence $\{T_m\}_{m<n}$ of infinite computable trees is said to be an *n-layer* if $T_m^{ext} \subseteq T_{m+1}$ for each $m < n-1$. This definition is essentially equivalent to the definition of n-layers of open balls, by Proposition 3. Let P be an n-layered co-c.e. closed set with an n-layer $\{T_m\}_{m<n}$, and Q be a co-c.e. closed set. Let Φ be a computable function from P to Q. As P is co-c.e. closed, we may safely assume that Φ is total. It suffices to show that the sequence $\{\Phi(T_m)\}_{m<n}$ of images of T_m's under Φ forms an n-layer of Q. Note that $\Phi(T_m)$ is computable for any $m \leq n$, by totality of Φ. Fix $m < n-1$. For each leaf ρ of $\Phi(T_m)$, we must have a leaf ρ^* of T_m with $\Phi(\rho^*) = \rho$. As $T_m \subseteq T_{m+1}^{ext}$, there are infinitely many nodes of T_{m+1} extending ρ^*. By weak König's lemma, T_{m+1} has an infinite path g extending ρ^*, and then g belongs to P, since $[T_{m+1}] \subseteq P$. Therefore, $\Phi(g) \in Q$ by our assumption that $\mathrm{dom}(\Phi)$ includes P. Then, $\Phi(T_{m+1})$ has a path $\Phi(g) \in Q$ extending $\Phi(\rho^*) = \rho$, i.e., $\rho \in \Phi(T_m)$ is extendible in $\Phi(T_{m+1})$. Hence, we have $\Phi(T_m) \subseteq (\Phi(T_{m+1}))^{ext}$, as desired. □

Definition 3. *Fix $P \subseteq X$. The* layer density *of a point $\alpha \in X$ on P is defined as* $\mathrm{density}_P(\alpha) = \inf\{\mathrm{density}(P \cap O) : \alpha \in O \in \mathbf{\Sigma}^0_1(X)\}$. *For $n \in \omega \cup \{\omega, \infty\}$ a point $\alpha \in X$ is an* n-layered accumulation point *of P if* $\mathrm{density}_P(\alpha) \geq n$.

Theorem 2. *Let $P, Q \subseteq 2^\omega$ be co-c.e. closed sets with no computable element. If a learnable function exists from P to Q, then* $\mathrm{density}(P) \leq \max\{\omega, \mathrm{density}(Q)\}$.

Proof. Fix an ∞-layered co-c.e. closed set $P \subseteq 2^\omega$ and a computable function $F : P \to 2^\omega$. By Blum-Blum Locking Lemma 1, there is a string σ extendible in $P^\heartsuit = \{\alpha \in P : \mathrm{density}_P(\alpha) = \mathrm{density}(P)\}$ such that $F \upharpoonright [\sigma]$ is computable, since $P^\heartsuit$ is nonempty and closed. Moreover, $\mathrm{density}(P^\heartsuit) = \mathrm{density}(P) = \infty$. The image of an ∞-layer by a computable function is again an ∞-layer. Therefore, $F(P)$ is ∞-layered. □

For elements a, b of a lattice L, we say that *a cups to b* if a is one-half of a witness of join-reducibility of b. For a bounded lattice L and $a \in L$, we also say that *a is cuppable in L* if a cups to $\max L$. We define preorders $\leq^1_1$ and $\leq^1_\omega$ on $\mathcal{P}(\omega^\omega)$ as follows: $P \leq^1_1 Q$ (resp. $P \leq^1_\omega Q$) if there is a partial computable (resp. learnable) function F on ω^ω such that $\mathrm{dom}(F) \supseteq P$ and $F(P) \subseteq Q$. The structures $\mathcal{P}(\omega^\omega)/\equiv^1_1$ and $\mathcal{P}(\omega^\omega)/\equiv^1_\omega$ form lattices, where the supremum in these lattices are given by $P \otimes Q = \{p \oplus q : (p,q) \in P \times Q\}$. The former lattice is called the *Medvedev lattice*, and the latter lattice is said to be the *degrees of nonlearnability* [6].

Theorem 3. *For each $n \in \omega \cup \{\infty\}$, let LD_n denote the set of all Medvedev degrees of n-layered co-c.e. closed sets in 2^ω. Then, the set LD_n is a principal prime ideal in LD_1, and every element of LD_{n+1} is noncuppable in LD_n.*

Moreover, LD_∞ *is a principal prime ideal in the degrees of nonlearnability of nonempty co-c.e. closed sets.*

Proof. See Cenzer et al. [4, Corollary 4.13]. Indeed, the top element of LD_n is the Medvedev degree of $\mathsf{PA}^{(n)}$, where PA denotes the set of all consistent complete theories extending Peano Arithmetic. For principality, by Higuchi-Kihara [6], $\mathsf{PA}^{(n+1)}$ is noncuppable in LD_n, i.e., $\mathsf{PA}^{(n+1)}$ does not cup to $\mathsf{PA}^{(n)}$. □

Fix a countable base $\mathfrak{O}$ of Cantor space 2^ω. A set $P \subseteq 2^\omega$ is *totally* ∞*-layered* if it is ∞-layered, and there exists a computable function $\mathfrak{B} : \mathfrak{O} \times \omega \to (\mathfrak{O}^\omega)^{<\omega}$ such that $\mathfrak{B}(U, n)$ forms an n-layer of $P \cap U$, whenever $P \cap U$ is ∞-layered.

Example 2. Fix a co-c.e. closed set $P = [T_P] \subseteq 2^\omega$. Then $P^\blacktriangledown$ denotes the set of all infinite paths through the tree consisting of strings of the form $\rho_0{}^\frown\tau(0)^\frown\rho_1{}^\frown\tau(1)$ ${}^\frown\rho_2{}^\frown \dots {}^\frown\rho_{|\tau|-1}{}^\frown\tau(|\tau|-1)^\frown\sigma$, where $\sigma, \tau \in T_P$ and each ρ_i is a leaf of T_P. Then, $P^\blacktriangledown$ is totally ∞-layered, and $(P^\blacktriangledown)^\heartsuit = \{\alpha \in P^\blacktriangledown : \mathrm{density}_{P^\blacktriangledown}(\alpha) = \mathrm{density}(P^\blacktriangledown)\}$ is co-c.e. closed.

Theorem 4. *If a totally* ∞*-layered set* P *has a co-c.e. closed subset* $P^\star$ *consisting of* ∞*-layered accumulation points, then* P *is noncuppable in the degrees of nonlearnability of co-c.e. closed subsets of* 2^ω*.*

Lemma 2. *Let* $C(X)$ *denote the space of all continuous functions on* X*. There exists a computable function* $\Xi : C(\omega^\omega) \times \mathcal{A}_-(2)^\omega \times (2^{<\omega})^\omega \times \omega^\omega \to \omega^\omega$ *such that, for any* $(f, H, (\sigma_i)_{i\in\omega}, \alpha) \in C(\omega^\omega) \times \mathcal{A}_-(2)^\omega \times (2^{<\omega})^\omega \times \omega^\omega$*, if the image of* $f|_{[\sigma_i]\otimes\{\alpha\}}$ *intersects with the product set* $H \subseteq 2^\omega$ *for every* $i \in \omega$*, then* $\Xi(f, H, (\sigma_i)_{i\in\omega}, \alpha)$ *is contained in* H*.*

Proof. Indeed, the proof of Cenzer et al. [4, Theorem 5.2] is uniform, where their theorem states that, if a co-c.e. closed set P is $\mathcal{B}$-dense for some infinite computable sequence $\mathcal{B} = \{[\sigma_i]\}_{i\in\omega}$ of intervals (i.e., P is not immune), then it does not cup to any separating class $H \in \mathcal{A}_-(2)^\omega$. In other words, if a computable function $f : P \otimes R \to H$ exists, then we have a computable function $\Xi : \omega^\omega \to \omega^\omega$ such that $\Xi(\alpha) \in H$ for any $\alpha \in R$. □

Proof (Theorem 4). Fix a learnable function $F = \lim_s F_s : P \otimes R \to \mathsf{PA}$. Note that $P^\star \otimes \{g\}$ is closed for any $g \in R$. Therefore, by Blum-Blum Locking Lemma 1, there must exist an extendible string ρ in $P^\star$ such that $G_\rho = F|_{(P^\star\cap[\rho])\otimes\{g\}}$ is computable. Then, we can find a sequence $\{\sigma_i^\rho\}_{i\in\omega}$ extending ρ such that $P^\star \cap [\sigma_i^\rho] \neq \emptyset$, since P is totally ∞-layered. Therefore, $\Xi(G_\rho, \mathsf{PA}, (\sigma_i^\rho)_{i\in\omega}, g)$ is contained in PA, where Ξ is a computable function in Lemma 2. From an input $g \in R$, one can learn a ρ^g such that $\rho^g \in P^\star$ and $\Gamma_s|_{\rho^g\otimes\{g\}} = \Gamma_{|\rho^g|}|_{\rho^g\otimes\{g\}}$ for any $s \geq |\rho^g|$, since the assertion $\Gamma_s|_Y = \Gamma_t|_Y$ is equivalent to the following: for any clopen set $[\sigma]$ and any $u \in [t, s]$, such that $\Gamma_t^{-1}(\{?\}) \cap Y \neq \emptyset$. Here recall that $\{?\}$ is a clopen set in $(\omega^\omega)_?$, and hence, $\Gamma_t^{-1}(\{?\})$ is c.e. open. Therefore, there is a $\Pi^0_1(g)$ statement characterizing ρ^g, uniformly in $g \in R$. Then, we have a learnable function $h = \lim_s h_s : R \to 2^\omega$ which maps g to such ρ^g. Define $\Delta_s(g) =?$ if $h_s(g) =?$, and $\Delta_s(g) = \Xi(G_{h_s(g)}, \mathsf{PA}, (\sigma_i^{h_s(g)})_{i\in\omega}, g)$ otherwise. It is easy to see that the learnable function $\Delta = \lim_s \Delta_s$ maps R into PA. □

3.2 Topological Games and Popperian Learnability

By Lewis-Shore-Sorbi [7], the initial segment $(\mathbf{0}, \mathbf{d}]$ below the Medvedev degree $\mathbf{d}$ of a dense set in ω^ω has no co-c.e. closed set. There are other density-like properties making *co-c.e.-free initial segments*:

For a set $S \subseteq X$, the two-players game $\mathfrak{G}_S$ is defined as follows: Each *play* is a decreasing sequence $\{U_n\}_{n\in\omega}$ of open sets with $S \cap U_n \neq \emptyset$. For a play $p = \{U_n\}_{n\in\omega}$, Player II *wins* on p if $S \cap \bigcap_n U_n \neq \emptyset$. Otherwise, Player I wins. If Player II has a winning strategy for the game $\mathfrak{G}_S$, then S is called *Choquet*.

$$\begin{array}{lcccccccc} \text{Player I:} & U_0 & & U_2 & & U_4 & & \dots & \\ & \supseteq & \supseteq & \supseteq & \supseteq & \supseteq & \supseteq & & \\ \text{Player II:} & & U_1 & & U_3 & & U_5 & & \dots \end{array}$$

Theorem 5. *Assume that a set $P \subseteq 2^\omega$ contains a Choquet subset $C \subseteq P$ whose closure has a dense subset of computable points. For any co-c.e. closed set $Q \subseteq 2^\omega$, if an e.P. learnable function exists from P to Q, then Q contains a computable element.*

Proof. Let $F :\subseteq \omega^\omega \to \omega^\omega$ be a partial learnable function. A partial computable function $f :\subseteq \omega^{<\omega} \to \omega^{<\omega} \cup \{?\}$ is said to be an *approximation of* F if:

- (?-goodness) $f(\sigma^-) \not\subseteq f(\sigma)$ occurs only when $? \in \{f(\sigma^-), f(\sigma)\}$;
- (Convergence) $F(x) = \lim_s f(x \upharpoonright s)$, for any $x \in \mathrm{dom}(F)$.

Fix a winning strategy ψ_{II} for Player II on the Choquet game $\mathfrak{G}_C$, a co-c.e. closed set $Q \subseteq 2^\omega$ with no computable element, and suppose that an e.P. learnable function $F : P \to Q$ exists. Fix also an approximation $f : \omega^{<\omega} \to \omega^{<\omega} \cup \{?\}$ of F. Choose any string τ_i with $[\tau_i] \cap C \neq \emptyset$. Since $\mathrm{cl}(C)$ has a dense subset of computable points, C is dense at a computable point $\beta_i \supset \tau_i$. Note that, if $f(\beta_i \upharpoonright n) \neq ?$ for any $n \geq |\tau_i|$, then $[f(\sigma)] \cap Q = \emptyset$ for some $\sigma \subset \beta_i$. Otherwise, since F is e.P., we have $\lim_n f(\beta_i \upharpoonright n) \in Q$. However, monotonicity of $\{f(\beta_i \upharpoonright n)\}_{|\tau_i| \leq n \in \omega}$ implies that $\lim_n f(\beta_i \upharpoonright n)$ is computable. This contradicts our assumption that Q contains no computable element. If $f(\sigma) \notin T_Q$ happens for some $\sigma \subset \beta_i$ extending τ_i, for any $\alpha \in C$ extending σ, we have $f(\sigma) \not\subseteq \lim_s f(\alpha \upharpoonright s) \in Q$, since $F(C) \subseteq Q$. Therefore, $f(\sigma^*) = ?$ must occur for some σ^* with $\tau_i \subset \sigma^* \subset \alpha$. Then, define $\psi_{\mathrm{I}}(\tau_i) = \sigma^*$. Player II extend it to $\tau_{i+1} = \psi_{\mathrm{II}}(\psi_{\mathrm{I}}(\tau_i))$. Eventually an infinite increasing sequence $\{\tau_i\}_{i\in\omega}$ is constructed, and then $h = \lim_i \tau_i \in C$ by the property of ψ_{II}. However, $\lim_n f(h \upharpoonright n)$ does not converge. Therefore, $h \notin \mathrm{dom}(F)$. ◻

Definition 4. *Fix a set $S \subseteq X$, and we consider the Choquet game $\mathfrak{G}_S$.*

1. *A function ψ is a* strategy *if, for a given open set b_n in X, $\psi(b_n)$ is an open subset of b_n, and $S \cap \psi(b_n) \neq \emptyset$ whenever $S \cap b_n \neq \emptyset$.*
2. *A function ψ is a* prestrategy *if, for a given previous move a_θ, $\psi(a_\theta)$ is a pair $\langle b_{\theta 0}, b_{\theta 1}\rangle$ of open sets with $b_{\theta 0} \cup b_{\theta 1} \subseteq a_\theta$, or $\psi(a_\theta) = \textsc{Resign}$, where we declare that $S \cap \textsc{Resign} = \emptyset$.*
3. *For a strategy ψ_{I} and a prestrategy ψ_{II},* the preplay $\psi_{\mathrm{I}} \otimes \psi_{\mathrm{II}}$ produced by ψ_{I} and ψ_{II} *is a collection $\langle a_{\langle\rangle}, b_{\theta j}, a_{\theta j}\rangle_{\theta\in 2^{<\omega}, j<2}$, where $a_{\langle\rangle} = \psi_{\mathrm{I}}(\langle\rangle)$, $\psi_{\mathrm{II}}(a_\theta) = \langle b_{\theta 0}, b_{\theta 1}\rangle$, and $a_{\theta j} = \psi_{\mathrm{I}}(b_{\theta j})$ for any $\theta \in 2^{<\omega}$, and $j < 2$.*

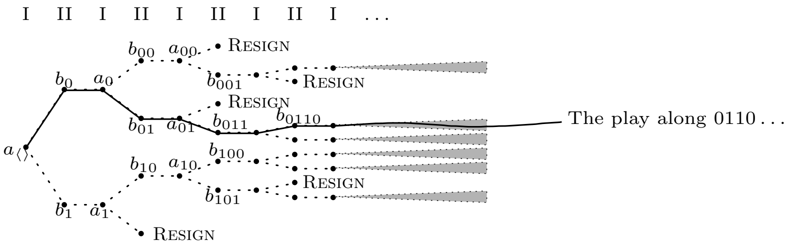

Fig. 1. A preplay on a given Choquet game

4. *For a preplay* $p = \langle a_{\langle\rangle}, b_{\theta j}, a_{\theta j}\rangle_{\theta\in 2^{<\omega}, j<2}$, the play of p along $h \in 2^\omega$ *is defined by the infinite sequence* $p|_h = \langle a_{\langle\rangle}, b_\theta, a_\theta\rangle_{\theta\subset h}$.
5. *The* play tree $\mathrm{Play}(\psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}})$ *of a preplay* $\psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}} = \langle a_{\langle\rangle}, b_{\theta j}, a_{\theta j}\rangle_{\theta\in 2^{<\omega}, j<2}$ *is defined by* $\mathrm{Play}(\psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}}) = \{\theta : (\forall\eta\subseteq\theta)\ b_\eta \neq \text{RESIGN}\}$. *For a partial preplay* $\pi \subset \psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}}$, *the play tree* $\mathrm{Play}(\pi)$ *is also defined in the same manner.*
6. *A prestrategy* ψ_{II} *for Player II is* winning *if, for every strategy* ψ_{I} *for Player I, Player II wins on the play of* $\psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}}$ *along any infinite path* h *through* $\mathrm{Play}(\psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}})$, *i.e.,* $S\cap\bigcap_n(\psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}}|_h)(n) \neq \emptyset$ *for any* $h \in [\mathrm{Play}(\psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}})]$.
7. *A function* ψ *is a* playful strategy *if it is a prestrategy, and the play tree* $\mathrm{Play}(\phi\otimes\psi)$ *has an infinite path for any strategy* ϕ.
8. *If Player II has a computable winning playful strategy for the game* $\mathfrak{G}_S$, *then* S *is called* PA-Choquet.

A partial computable function $\beta : \omega^{<\omega} \to \omega^\omega$ is a *dense choice of computable points in* C if $C\cap[\sigma]$ is dense at the point $\beta(\sigma)$, whenever $C\cap[\sigma]$ is nonempty.

Theorem 6. *Assume that a set* $P \subseteq 2^\omega$ *contains a* PA*-Choquet subset* $C \subseteq P$ *whose closure has a dense choice of computable points. For any co-c.e. closed set* $Q \subseteq 2^\omega$ *and any* $R \subseteq \omega^\omega$, *if an e.P. learnable function exists from* $P\otimes R$ *to* Q, *then an e.P. learnable function exists from* R *to* Q.

Proof. Fix a computable winning playful strategy ψ_{II} for the player II on the Choquet game $\mathfrak{G}_C$, a co-c.e. closed set $Q \subseteq 2^\omega$, and an e.P. learnable function $F : P\otimes R \to Q$ with an approximation $f : \omega^{<\omega} \to \omega^{<\omega}\cup\{?\}$. Let β be a dense choice of computable points in C. Fix $g \in R$.

Strategies S^g_θ. We introduce a strategy S^g_θ for each $\theta \in 2^{<\omega}$. There are four states for strategies, ACTIVE, CHANGED, REFUTED, and RESIGNED. First we declare the root strategy $S^g_{\langle\rangle}$ to be ACTIVE. Assume that, on a partial play on the Choquet game $\mathfrak{G}_C$, the θ-th move τ^g_θ of ψ_{II} is given, S^g_θ is ACTIVE, and there is no ACTIVE strategy S^g_κ for $\kappa \subsetneq \theta$. We determine the state of the θ-th strategy S^g_θ as follows:

- S^g_θ is CHANGED if $f(\sigma\oplus g) =?$ for some $\tau^g_\theta \subset \sigma \subset \beta(\tau^g_\theta)$.
- S^g_θ is REFUTED if $f(\sigma\oplus g) \notin T_Q$ for some $\tau^g_\theta \subset \sigma \subset \beta(\tau^g_\theta)$.
- S^g_θ is RESIGNED when we find that θ does not extend to an infinite path through the play tree $\mathrm{Play}(\psi_{\mathrm{I}}\otimes\psi_{\mathrm{II}})$ of the winning strategy ψ_{II}.

If S^g_θ is declared to be CHANGED, or RESIGNED, then we withdraw the previous declaration that S^g_θ is ACTIVE, and close the strategy S^g_θ.

Play on Choquet Game $\mathfrak{G}_C$. Now we determine the next move of Player I, i.e., define $\psi_{\rm I}(\tau^g_\theta)$. If S^g_θ is REFUTED or RESIGNED, then Player I takes no action. If S^g_θ is CHANGED, then Player I chooses the least σ such that S^g_θ is refuted at σ, and put $\psi_{\rm I}(\tau^g_\theta) = \sigma$. Then, by using the winning strategy $\psi_{\rm II}$, Player II chooses the $(\theta 0)$-th move $\tau^g_{\theta 0}$ and the $(\theta 1)$-th move $\tau^g_{\theta 1}$, from the partial play $\psi_{\rm I}(\tau^g_\theta)$, i.e., $\psi_{\rm II}(\psi_{\rm I}(\tau^g_\theta)) = \langle \tau^g_{\theta 0}, \tau^g_{\theta 1}\rangle$, and declare that the strategies $S^g_{\theta 0}$ and $S^g_{\theta 1}$ are ACTIVE. Note that τ^g_θ and the state of S^g_θ at each stage are partial computable uniformly in θ and g, since $\psi_{\rm II}$ and β are computable.

Observation. For any $g \in R$, consider the following binary tree V^g consisting of all binary strings $\theta \in 2^{<\omega}$ such that S^g_θ is declared to be ACTIVE at some stage. Claim that V^g has no infinite path. If V^g has an infinite path h, then f must outputs ? infinitely often along $p_h = \bigcup_{\theta \subset h} \tau_\theta$. However, p_h is constructed along the winning strategy $\psi_{\rm II}$, and p_h is an infinite path through the play tree $\mathrm{Play}(\psi_{\rm I} \otimes \psi_{\rm II})$, since no substring of h is RESIGNED. As $\psi_{\rm II}$ is winning, p_h must belong to $C \subseteq P$. It implies that $F(p_h \oplus g) = \lim_s f(p_h \oplus g \upharpoonright s)$ does not converges, and note that $p_h \oplus g \in C \otimes \{g\} \subseteq P \otimes R$, This contradicts our assumption that the domain of F includes $P \otimes R$.

Thus, at some stage, all declarations of strategies on V^g are determined. Moreover, each leaf of V^g which is not assigned RESIGN by $\psi_{\rm II}$ must be declared to be ACTIVE at almost all stages. Because C is dense at $\beta(\tau^g_\rho)$ for each leaf $\rho \in V^g$ which is not declared to be RESIGNED, and then $\lim_s f(\beta(\tau^g_\rho) \oplus g \upharpoonright s)$ is total, since $F = \lim f$ is e.P., and each leaf $\rho \in V^g$ must not be declared to be CHANGED. In particular, $\lim_s f(\beta(\tau^g_\rho) \oplus g \upharpoonright s) \in Q$.

Learning Procedure. We construct an e.P. learnable function $G : R \to Q$. The learner $G(g)$ tries to find an ACTIVE leaf ρ of V^g at each stage s, and set $G(g) = F(\beta(\tau^g_\theta) \oplus g)$. Each time his guess on an eventually ACTIVE leaf of V^g is changed, an approximation of G returns ?. If g is contained in R, then by finiteness of V^g, an approximation of $G(g)$ eventually finds an ACTIVE leaf of V^g. If $g \notin R$, then $G(g)$ may yet fail to find an ACTIVE leaf of V^g. But then its approximation returns ? infinitely often. Otherwise, $G(g)$ is defined to be $F(\beta(\tau^g_\theta) \oplus g)$, and then it is e.P., since F is e.P. By the previous observation, the e.P. learnable function G maps R into Q as desired. □

Definition 5 (Higuchi-Kihara [6]). *Fix $\sigma \in \omega^{<\omega}$, and $i \in \omega$. Then* the i-th projection of σ *is inductively defined as follows.*

$$\mathtt{pr}_i(\langle\rangle) = \langle\rangle, \qquad \mathtt{pr}_i(\sigma) = \begin{cases} \mathtt{pr}_i(\sigma^-)^\frown n, & \text{if } \sigma = \sigma^{-\frown}\langle i, n\rangle, \\ \mathtt{pr}_i(\sigma^-), & \text{otherwise.} \end{cases}$$

Furthermore, the projection of $x \in \omega^\omega$ is defined to be $\mathtt{pr}_i(x) = \lim_n \mathtt{pr}_i(x \upharpoonright n)$.

Theorem 7. *For every co-c.e. closed set $P \subseteq 2^\omega$, for each $k \geq 2$, the set* $\mathrm{TEAM}_k\mathrm{LEARNING}(P) = \{x \in \omega^\omega : (\exists i < k)\ \mathtt{pr}_i(x) \in P^{(\infty)}\}$ *is a Σ^0_3 subset of 2^ω which has the same Turing upward closure as P, and has a* PA*-Choquet subset whose closure has a dense choice of computable points.*

Proof. Set $S = \textsc{Team}_2\textsc{Learning}(P)$. Straightforwardly, we can check that S is Σ^0_3, and it has the same Turing upward closure as P. Consider the following set:

$$C = \{x \in \omega^\omega : \mathtt{pr}_0(x) \in P^{(\infty)} \ \& \ (\forall n \in \omega)\ \mathtt{pr}_1(x \upharpoonright n) \in T_P^{ext}\}.$$

Clearly, C is a subset of S. To construct a dense choice β of computable points in the closure of C, we fix a leaf of T_P. Given σ, if it has a nonempty intersection with C, then $\mathtt{pr}_0(\sigma)$ must be of the form $\rho_0{}^\frown\rho_1{}^\frown\dots{}^\frown\rho_n{}^\frown\tau$, where ρ_i is a leaf of T_P for each $i \leq n$, and τ is a node of T_P. By a uniformly computable way, we can calculate the position of a leaf $\tau^\frown\eta$ of T_P. Then, define $\beta(\sigma)$ as follows:

$$\beta(\sigma) = \sigma^\frown(0^{|\eta|} \oplus \eta)^\frown(0^{|\rho|} \oplus \rho)^\frown(0^{|\rho|} \oplus \rho)^\frown\dots{}^\frown(0^{|\rho|} \oplus \rho)^\frown(0^{|\rho|} \oplus \rho)^\frown\dots$$

Here, $0^{|\alpha|} \oplus \alpha$ denotes the string $\langle 0, \alpha(0), 0, \alpha(1), \dots, 0, \alpha(|\alpha|-1)\rangle$. Clearly, $\beta(\sigma)$ is contained in the closure of C.

Now we construct a strategy ψ for Player II on Choquet game $\mathfrak{G}_C$ as follows: Given $a_\theta \in \omega^{<\omega}$, the θ-th move of Player I, first check whether $\mathtt{pr}_1(a_\theta)$ has an extension in T_P of length $\max\{|\mathtt{pr}_1(a_\theta)|, |\theta|\}$ or not. If not (it is possible because of the past moves by Player II), Player II resigns the game $\mathfrak{G}_C$, i.e., $\psi_{\mathrm{II}}(a_\theta) = \textsc{Resign}$. Otherwise, when $|\mathtt{pr}_1(a_\theta)| > |\theta|$, Player II does not act, i.e., $\psi_{\mathrm{II}}(a_\theta) = \langle a_\theta, a_\theta\rangle$. If $\mathtt{pr}_1(a_\theta) \leq |\theta|$, then Player II returns $\psi_{\mathrm{II}}(a_\theta) = \langle a_\theta{}^\frown\langle 1, 0\rangle, a_\theta{}^\frown\langle 1, 1\rangle\rangle$. By our construction of the strategy ψ_{II}, for every $\psi_{\mathrm{I}} \otimes \psi_{\mathrm{II}}|_h$ along any infinite path h through the play tree $\mathrm{Play}(\psi_{\mathrm{I}} \otimes \psi_{\mathrm{II}})$, the 1-st projection of $\bigcap_n \psi_{\mathrm{I}} \otimes \psi_{\mathrm{II}}|_h$ must be contained in P. Therefore, $\bigcap_n \psi_{\mathrm{I}} \otimes \psi_{\mathrm{II}}|_h$ is contained in C. Moreover, P is equal to the set of all infinite paths through $\mathrm{Play}(\psi_{\mathrm{I}} \otimes \psi_{\mathrm{II}})$. Consequently, ψ_{II} is a winning playful strategy of Player II. □

References

1. Stephen Binns. Hyperimmunity in $2^{\mathbb{N}}$. *Notre Dame Journal of Formal Logic*, 48(2):293–316, 2007.
2. Vasco Brattka and Guido Gherardi. Effective choice and boundedness principles in computable analysis. *Bulletin of Symbolic Logic*, 17(1):73–117, 2011.
3. Vasco Brattka and Gero Presser. Computability on subsets of metric spaces. *Theor. Comput. Sci.*, 305(1-3):43–76, 2003.
4. Douglas Cenzer, Takayuki Kihara, Rebecca Weber, and Guohua Wu. Immunity and non-cupping for closed sets. *Tbilisi Math. J.*, 2:77–94, 2009.
5. Osvald Demuth and Antonín Kučera. Remarks on 1-genericity, semigenericity and related concepts. *Comment. Math. Univ. Carolinae*, 28:85–94, 1987.
6. Kojiro Higuchi and Takayuki Kihara. Inside the Muchnik degrees: Discontinuity, learnability, and constructivism. preprint.
7. Andrew E. M. Lewis, Richard A. Shore, and Andrea Sorbi. Topological aspects of the Medvedev lattice. *Arch. Math. Log.*, 50(3-4):319–340, 2011.
8. Robert I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic. Springer, Heidelberg, 1987. XVIII+437 pages.
9. Klaus Weihrauch. *Computable Analysis: An Introduction.* Texts in Theoretical Computer Science. Springer, 2000. 285 pages.
10. Martin Ziegler. Real computation with least discrete advice: A complexity theory of nonuniform computability with applications to effective linear algebra. *Ann. Pure Appl. Log.*, to appear.