NOTES ON $\forall\exists!$-CONSERVATION

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ABSTRACT. Some $\forall\exists!$-conservation results.

Definition 0.1. A theory Γ is *AEU-conservative* over another theory Λ if and only if

$$\Gamma + \Lambda \vdash \psi \Leftrightarrow \Lambda \vdash \psi$$

for every Π^1_2-sentence ψ of the form $\forall X \exists! Y \varphi$ where φ is arithmetic.

The general approach to obtain AEU-conservation is as following: given any (countable) model $\mathcal{M} = (M, \mathcal{S}_0) \models \Lambda$, build $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ such that $\mathcal{S}_0 = \mathcal{S}_1 \cap \mathcal{S}_2$, $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{S}_3$ and $(M, \mathcal{S}_i) \models \Gamma + \Lambda$ for $1 \leq i \leq 3$.

1. COH IS AEU-CONSERVATIVE OVER RCA_0

Theorem 1.1. COH *is AEU-conservative over* RCA_0.

Proof. Fix a countable $\mathcal{M} = (M, \mathcal{S}_0) \models \mathrm{RCA}_0$.

By Mathias forcing, it is easy to construct an M-infinite G_0 such that $\mathcal{M}[G_0] \models \mathrm{RCA}_0$ and G_0 is $\mathcal{M}$-cohesive, i.e., for every $X \in \mathcal{S}_0$ either $G_0 - X$ or $G_0 - (M - X)$ is M-finite.

Suppose that we have constructed G_i for $i < 2k+1$ such that

- $\mathcal{M}[\bigoplus_{i<2k+1} G_i] \models \mathrm{RCA}_0$,
- G_{2j} (G_{2j+1}) is cohesive over $\mathcal{M}[\bigoplus_{j'<j} G_{2j'}]$ ($\mathcal{M}[\bigoplus_{j'<j} G_{2j'+1}]$),
- $\mathcal{M}[\bigoplus_{j<k+1} G_{2j}] \cap \mathcal{M}[\bigoplus_{j<k} G_{2j+1}] = \mathcal{M}$.

We construct G_{2k+1} by constructing a sequence of Mathias conditions $((\sigma_n, X_n) : n \in \omega)$ such that

1. $\sigma_n \in M$ and $X_n \in \mathcal{M}[\bigoplus_{j<k} G_{2j+1}]$,
2. $\sigma_n \subset \sigma_{n+1}$ and $(\sigma_{n+1}, X_{n+1}) \leq_M (\sigma_n, X_n)$,
3. for each $X \in \mathcal{M}[\bigoplus_{j<k} G_{2j+1}]$ there exists n such that either $X_n \subset X$ or $X_n \subset M - X$,
4. for each $e \in M$ there exists n such that either $\Phi_e(\bigoplus_{j<k+1} G_{2j}) \neq \Phi_e(\bigoplus_{j<k} G_{2j+1} \oplus \sigma_n)$ or $\Phi_e(\bigoplus_{j<k} G_{2j+1} \oplus G) \leq_T \bigoplus_{j<k} G_{2j+1}$ for every G satisfying (σ_n, X_n),
5. for each Σ_1 formula $\varphi(x)$ there exists n such that

$$\mathcal{M}[\bigoplus_{i<2k+1} G_i] \models (\sigma_n, X_n) \Vdash I\varphi.$$

(1) is automatic. (2) and (3) are easy. (4) can be obtained by splitting. (5) can be done as the proof that COH is Π^1_1-conservative over RCA_0, because that $\mathcal{M}[\bigoplus_{i<2k+1} G_i] \models \mathrm{RCA}_0$.

As soon as we have G_{2k+1}, we can construct G_{2k+2} with similar properties. Eventually we have a sequence $(G_n : n \in \omega)$ such that

- $\mathcal{M}[G_0, G_1, \ldots, G_n, \ldots] \models \mathrm{RCA}_0$,
- G_{2k} (G_{2k+1}) is cohesive over $\mathcal{M}[\bigoplus_{j<k} G_{2j}]$ ($\mathcal{M}[\bigoplus_{j<k} G_{2j+1}]$),
- $\mathcal{M}[G_0, G_2, \ldots, G_{2k}, \ldots] \cap \mathcal{M}[G_1, G_3, \ldots, G_{2k+1}, \ldots] = \mathcal{M}$.

By the proof that COH is Π^1_1-conservative over RCA_0, there exists $(M, \mathcal{S})$ such that $\mathcal{M}[G_0, G_1, \ldots, G_n, \ldots] \subseteq (M, \mathcal{S}) \models \mathrm{COH}$. $\square$