Let $\mathcal{O}$ be Kleene's system of ordinal notations. Every $a \in \mathcal{O}$ automatically produces a fundamental sequence for the corresponding ordinal. Therefore, the notation $a \in \mathcal{O}$ automatically generates the $a$-th fast growing function $f_a$ in a straightforward manner. Note that $f_a$ depends on the notation $a$, but not on the ordinal $|a|_{\mathcal{O}}$.

Put $\alpha_0$ as the notation for the ordinal 0, and

$$\alpha_{n+1} = \min\{m \in \mathcal{O} \mid m > \alpha_n \text{ and } |m|_{\mathcal{O}} > |\alpha_n|_{\mathcal{O}}\}.$$

The sequence $(|\alpha_n|_{\mathcal{O}})_{n<\omega}$ is clearly a fundamental sequence for $\omega_1^{\mathrm{CK}}$. Then, we define

$$f_{\omega_1^{\mathrm{CK}}}(n) = f_{\alpha_n}(n).$$

Even if we use the above natural fundamental sequence $(\alpha_n)_n$ (or $(\gamma_n)_n$), we show that it is possible that $f_{\omega_1^{\mathrm{CK}}}$ may be very slow growing. This is due to the basic observation that the definition of $\mathcal{O}$ (hence $f_{\omega_1^{\mathrm{CK}}}$) heavily depends on a given *numbering of partial computable functions.*

**Proposition 1.** *There is an admissible numbering of partial computable functions such that $f_{\omega_1^{\mathrm{CK}}}$ is dominated by the $(\omega+3)$rd fast growing function $f_{\omega+3}$.*

*Proof.* Let $\Phi$ be a canonical admissible numbering of partial computable functions. One can assume that for almost all $e$ there is $d$ with $e < d < 2e$ such that $\Phi_d(0) = e$ and $\Phi_d(n+1) = 2^{\Phi_d(n)}$. In other words, if $e$ is a code of an ordinal $\alpha$, then $\Phi_d(n)$ is a code of $\alpha + n$; hence $3 \cdot 5^d$ is a code for $\alpha + \omega$. For example, in a usual programming language, there is a constant $c$ (independent of $e$) such that the length $|d| \approx \log_2(d)$ of such a program $d$ is bounded by $|e| + c$.

We construct a new numbering $\Psi$ by defining

$$\Psi_{2e}(n) = \begin{cases} 2 \uparrow\uparrow n, & \text{if } n \leqslant f^{(3)}_{\omega+3}(3 \cdot 5^{2e} + 3), \\ \Phi_e, & \text{otherwise.} \end{cases}$$

and $\Psi_{2e+1} = \Phi_e$. It is easy to check that $\Psi$ is admissible. We now assume that the new numbering $\Psi$ is used to define $\mathcal{O}$ (hence $(\alpha_n)_n$).

We first claim that $\alpha_n \neq 3 \cdot 5^{2e+1}$ for any $n, e$. If $3 \cdot 5^{2e+1} \notin \mathcal{O}$, the claim trivially holds, so we assume $3 \cdot 5^{2e+1} \in \mathcal{O}$. Then, the function $\Psi_{2e+1}$ is increasing w.r.t. $<_{\mathcal{O}}$; hence $2 \uparrow\uparrow n \leqslant_{\mathcal{O}} \Psi_{2e+1}(n)$ for any $n$. This implies that $\Psi_{2e}$ is also increasing w.r.t. $<_{\mathcal{O}}$, which means $3 \cdot 5^{2e} \in \mathcal{O}$. Since $\Psi_{2e}(n) = \Psi_{2e+1}(n)$ holds for almost all $n$, we have $\{a : a <_{\mathcal{O}} 3 \cdot 5^{2e}\} = \{a : a <_{\mathcal{O}} 3 \cdot 5^{2e+1}\}$. Thus, $3 \cdot 5^{2e+1}$ cannot be equal to the least $m$ such that $\alpha_{n-1} < m$ and $|\alpha_{n-1}|_{\mathcal{O}} < |m|_{\mathcal{O}}$ (or $\alpha_{n-1} <_{\mathcal{O}} m$).

Assume that $\alpha_n = 2^b$ for some $b$ and $n > 0$. We claim that $\alpha_{n-1} = b$. First, $|\alpha_{n-1}|_{\mathcal{O}} < |\alpha_n|_{\mathcal{O}} = |b|_{\mathcal{O}} + 1$ implies $|\alpha_{n-1}| \leqslant |b|_{\mathcal{O}}$. If $\alpha_{n-1} > b$ then $|\alpha_{n-2}|_{\mathcal{O}} < |\alpha_{n-1}|_{\mathcal{O}} \leqslant |b|_{\mathcal{O}}$ is chosen as $\alpha_{n-1}$, a contradiction. Thus, we can assume $\alpha_{n-1} \leqslant b$. Note that $|\alpha_{n-1}|_{\mathcal{O}} < |b|_{\mathcal{O}}$ is impossible; otherwise $b$ must be chosen as $\alpha_n$ by our assumption $\alpha_{n-1} \leqslant b$. Hence, $|\alpha_{n-1}|_{\mathcal{O}} = |b|_{\mathcal{O}}$. If $\alpha_{n-1} < b$, we have $2^{\alpha_{n-1}} < 2^b$; hence $2^{\alpha_{n-1}}$ is chosen as $\alpha_n$. As a consequence, we obtain $\alpha_{n-1} = b$.

Hence, if $\alpha_n$ codes an ordinal $\lambda + p$ for some limit $\lambda$ and finite $p$, then we have a sequence $\alpha_{n-p} <_{\mathcal{O}} \alpha_{n-p+1} <_{\mathcal{O}} \cdots <_{\mathcal{O}} \alpha_n$ which codes $\lambda < \lambda + 1 < \cdots < \lambda + p$. Put $m = n - p$. Then $\alpha_m$ is of the form $3 \cdot 5^{2e}$. If $m$ is sufficiently large, as mentioned above, there is $d$ with $e < d < 2e$ such that $3 \cdot 5^{2d}$ codes the ordinal $\lambda + \omega$. Then

$3 \cdot 5^{2d} < 3 \cdot 5^{2e} < \exp_2^{(4)}(e) := 2^{2^{2^{2^e}}}$; hence, the number $3 \cdot 5^{2d}$ coding $\lambda + \omega$ has higher priority than the number $\exp_2^{(4)}(e)$ coding $\lambda + 4$. In other words, we must have $p \leqslant 3$.

We now calculate $f_{\omega_1^{\mathrm{CK}}}(n)$. As observed above, for almost all $n$, $\alpha_n$ is of the form $\exp_2^{(p)}(3 \cdot 5^{2e})$ for some $p \leqslant 3$. If $p = 0$, as $\alpha_n$ is $<$-increasing, we clearly have $n \leqslant 3 \cdot 5^{2e}$. Hence, for any $p \leqslant 3$, we have $n \leqslant 3 \cdot 5^{2e} + 3$. If $p = 0$, our definition of $\Psi_{2e}$ ensures that $f_{\alpha_n}(m) = f_m(m) = f_\omega(m)$ for any $m \leqslant f_{\omega+3}^{(3)}(a+3)$, where $a = 3 \cdot 5^{2e}$. If $m \leqslant f_{\omega+3}^{(2)}(a+3)$, then

$$f_\omega^{(m)}\left(f_{\omega+3}^{(2)}(a+3)\right) \leqslant f_{\omega+1}\left(f_{\omega+3}^{(2)}(a+3)\right) \leqslant f_{\omega+3}^{(2)}(a+3)$$

and thus, if $p = 1$, then $f_{\alpha_n}(m) = f_a^{(m)}(m) = f_\omega^{(m)}(m) = f_{\omega+1}(m)$ for any such $m$. Similarly, one can observe that, if $p = 2$, then $f_{\alpha_n}(m) = f_{\omega+2}(m)$ for any $m \leqslant f_{\omega+3}(a+3)$, and if $p = 3$, then $f_{\alpha_n}(m) = f_{\omega+3}(m)$ for any $m \leqslant a + 3$. In any case, we have $f_{\alpha_n}(n) \leqslant f_{\omega+3}(n)$ since $n \leqslant a + 3$ as observed above. Consequently, $f_{\omega_1^{\mathrm{CK}}}(n) = f_{\alpha_n}(n) \leqslant f_{\omega+3}(n)$ for almost all $n$, that is, $f_{\omega_1^{\mathrm{CK}}}$ is dominated by $f_{\omega+3}$. $\square$